

VARIATIONAL PRINCIPLES IN NONLINEAR VISCOELASTICITY

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Abstract—A general variational principle is presented for the dynamic behavior of nonlinearly viscoelastic solids. Several alternate principles are also presented.

1. INTRODUCTION

HISTORICALLY, variational principles have represented an important part of theoretical mechanics. The classical theorem of minimum potential energy, for example, provides an alternative approach to the statics of elastic bodies and a basis for the study of stability. The concept of potential energy of elastic bodies, together with the notion of complementary energy, Reissner's principle, Hamilton's principle and numerous of their variants, provide bases for approximations in a wide realm of problems in mechanics. In recent times, variational formulations of problems in linear elastodynamics and linear viscoelasticity were developed by Gurtin [1, 2] and Leitman [3] and Nickell and Sackman [4] developed a variational statement of the equations governing linear thermoelasticity. A general approach for the development of variational principles of problems in linear continuum mechanics was presented by Sandhu and Pister [5, 6]. They employ a generalization of Mikhlin's theory for linear operators [7], in which the bilinear mapping suggested by Gurtin [1, 2] is used. References to other related works can be found in the papers cited.

In the present paper, we develop general variational principles governing the equations of nonlinearly viscoelastic bodies. We employ a generalization of a theorem presented by Vainberg [8] which reduces to that of Sandhu and Pister [5] and Mikhlin [7] as special cases. We outline the essential features of variational methods for nonlinear potential operators in the following section and Section 3 of the paper is devoted to a brief review of the governing equations of nonlinear viscoelasticity. Here we retain a significant degree of generality by assuming only that the stress tensor and the internal dissipation are known as functionals of the deformation gradients (or strains) and their histories. The general variational principle for nonlinear viscoelastic bodies is derived in Section 4 and a number of alternative principles are cited in Section 5. In particular, we demonstrate that, under appropriate additional assumptions, existing variational principles for problems in linear viscoelasticity [2, 3, 5, 6] can be precipitated from the general principle developed herein.

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2. SOME MATHEMATICAL PRELIMINARIES

Nonlinear operators on Banach spaces

Consider an equation of the form

$$\mathcal{P}(u) = f \tag{2.1}$$

where \mathcal{P} is a nonlinear operator defined on a dense set $\Omega \subseteq \mathcal{V}$, \mathcal{V} being a real Banach space, $u = u(\mathbf{x}, t)$ is an element of Ω , the domain of \mathcal{P} ; and $f = f(\mathbf{x}, t) \in \mathcal{V}$. Here the domain of \mathcal{P} consists of functions in the \mathcal{E}^4 -space defined by the cartesian product of the closure $\bar{\mathcal{R}}$ of an open, bounded, connected region \mathcal{R} in the three-dimensional Euclidean space \mathcal{E}^3 and the time interval $(-\infty, \infty)$, denoted $\bar{\mathcal{R}} \times (-\infty, \infty)[\mathbf{x} \in \bar{\mathcal{R}}, t \in (-\infty, \infty)]$. Equation (2.1) is to be satisfied in the interior of \mathcal{R} and on the boundary $\partial\mathcal{R}$, we must impose certain boundary conditions $\mathcal{L}(u) = g$, where \mathcal{L} is also a nonlinear operator.

Our primary purpose here is to construct a variation statement of (2.1) for the case in which it corresponds to the equations governing the nonlinear theory of viscoelasticity; that is, we wish to find a functional $K(u)$, which assumes a stationary value at the solutions of (2.1). Towards this end, we cite certain definitions and theorems concerned with potential operators. For further details, the monograph of Vainberg [8] can be consulted.

Derivatives of operators

Let \mathcal{V} be a real Banach space (i.e. a complete normed linear space whose elements can be multiplied by real numbers) and let \mathcal{P} be an operator that carries elements of \mathcal{V} into another Banach space \mathcal{V}' . The operator \mathcal{P} with domain in Ω is called continuous at $u_0 \in \mathcal{V}$, if for any sequence $\{u^n\}$ which converges in the norm to u_0 in the sense that $\lim_{n \rightarrow \infty} \|u^n - u_0\| = 0$, the sequence $\{\mathcal{P}(u^n)\}$ converges to $\mathcal{P}(u_0)$ i.e. $\lim_{n \rightarrow \infty} \|\mathcal{P}(u^n) - \mathcal{P}(u_0)\| = 0$.

The Gateaux differential of $\mathcal{P}(u)$, denoted $G\mathcal{P}(u, h)$ and the Frechet differential of $\mathcal{P}(u)$, denoted $\delta\mathcal{P}(u, h)$ are defined by

$$\lim_{\alpha \rightarrow 0} \left\| \frac{1}{\alpha} [\mathcal{P}(u + \alpha h) - \mathcal{P}(u)] - G\mathcal{P}(u, h) \right\| = 0 \tag{2.2}$$

and

$$\lim_{\|h\| \rightarrow 0} \|\mathcal{P}(u + h) - \mathcal{P}(u) - \delta\mathcal{P}(u, h)\| = 0 \tag{2.3}$$

where h is an arbitrary element in \mathcal{V} . The linear operators $\dot{\mathcal{P}}(u)(h)$ and $\mathcal{P}'(u)(h)$, where $\dot{\mathcal{P}}(u)(h) = G\mathcal{P}(u, h)$ and $\mathcal{P}'(u)(h) = \delta\mathcal{P}(u, h)$ are called Gateaux and Frechet derivatives of \mathcal{P} at u , respectively. Since it can be shown that if $\mathcal{P}(u)$ exists and is continuous in the neighborhood of u , then $G\mathcal{P}(u, h)$ exists and is identically the same as $\delta\mathcal{P}(u, h)$, we shall use the definitions and notations of (2.2) and (2.3) interchangeably, i.e. we henceforth assume the existence and continuity of $\mathcal{P}(u)$.

The gradient of a functional and potential operators

Suppose that a functional $K(u)$ has a linear Gateaux differential $\delta K(u, h)$ on some set $\Omega \subset \mathcal{V}$, i.e. for some $u \in \Omega$ and every $h \in \mathcal{V}$. We compute

$$\delta K(u, h) = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [K(u + \alpha h) - K(u)] \tag{2.4}$$

or equivalently

$$\delta K(u, h) = \frac{\partial}{\partial \alpha} K(u + \alpha h)|_{\alpha=0}. \tag{2.5}$$

Then $\delta K(u, h)$ describes a linear functional on \mathcal{V} and thereby identifies a mapping from Ω into \mathcal{V}^* , the conjugate of \mathcal{V} . This mapping, which we choose to identify by $\mathcal{P}(u)$, is called the gradient of functional $K(u)$ and is denoted $\text{grad } K(u)$. Evidentially $\mathcal{P}(u)$ defines a mapping from Ω into \mathcal{V}^* . Furthermore, $\delta K(u, h)$ is linear in h and $\text{grad } K(u)$ can be defined so that $\delta K(u, h)$ is also linear in $\mathcal{P}(u)$. Thus, we state that the operator $\mathcal{P}(u)$ defined by the formula

$$\langle \mathcal{P}(u), h \rangle = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [K(u + \alpha h) - K(u)] \tag{2.6}$$

for any $h \in \mathcal{V}$, is called the gradient of $K(u)$ and we write $\mathcal{P}(u) = \text{grad } K(u)$.

Potential operators

An operator $\mathcal{P}: \mathcal{V} \rightarrow \mathcal{V}^*$ is said to be a potential operator on some set $\Omega \subset \mathcal{V}$, if there exists a functional $K(u)$ such that $\text{grad } K(u) = \mathcal{P}(u)$ for every $u \in \Omega$.

It is shown by Vainberg [8] that a necessary and sufficient condition for an operator $\mathcal{P}: \mathcal{V} \rightarrow \mathcal{V}^*$ to be potential in $\mathcal{N}(u_0, r)$ is that the bilinear functional $\langle \delta \mathcal{P}(u, h_1), h_2 \rangle$ be symmetric in the sense that

$$\langle \delta \mathcal{P}(u, h_1), h_2 \rangle = \langle \delta \mathcal{P}(u, h_2), h_1 \rangle \tag{2.7}$$

for every $h_1, h_2 \in \mathcal{V}$ and $u \in \mathcal{N}(u_0, r)$ where $\langle \delta \mathcal{P}(u, h_1), h_2 \rangle$ is assumed to be continuous on $\mathcal{N}(u_0, r)$.

Thus, if \mathcal{P} in (2.1) is potential, solutions of (2.1) will be critical points of some functional $K(u)$. To generate such a functional of nonlinear operator equations such as (2.1) is answered by a fundamental theorem proved by Vainberg [8].

Theorem 1

Let $\mathcal{P}: \mathcal{V} \rightarrow \mathcal{V}^*$ be an operator, not necessarily linear, that is potential on $\mathcal{N}(u_0, r)$. Then there exists a unique functional $K(u)$, whose gradient is $\mathcal{P}(u)$, which is given by

$$K(u) = \int_0^1 \langle \mathcal{P}(u_0 + s(u - u_0)), (u - u_0) \rangle ds + K_0 \tag{2.8}$$

wherein $K_0 = K(u_0)$ is a constant, and s a real parameter.

Equation (2.8) provides a general method for determining variational statements of nonlinear equations of the form (2.1) provided $\mathcal{P}(\cdot)$ is a potential operator. If it is not potential [i.e. if (2.7) is not satisfied], then \mathcal{P} can be transformed into a potential operator by using any of the schemes described in [9].

3. THE GOVERNING FIELD EQUATIONS

The field equations governing the isothermal behavior of a viscoelastic medium are stated in this section. Let \bar{V} be a closed region of a three-dimensional Euclidean space occupied by a continuous body B whose interior we denote by V and the boundary \mathcal{B} .

Let $\bar{V} \times (-\infty, \infty)$ denote the domain of definition for all functions of position \mathbf{x} and time t . Furthermore, let $\mathbf{u}(\mathbf{x}, t)$, $\boldsymbol{\gamma}(\mathbf{x}, t)$, $\boldsymbol{\sigma}(\mathbf{x}, t)$, $\mathbf{F}(\mathbf{x}, t)$ and $\hat{\sigma}(\mathbf{x}, t)$ denote, respectively, the displacement vector, the strain tensor, the stress tensor, the body force vector and the internal dissipation, all defined for $(\mathbf{x}, t) \in \bar{V} \times (-\infty, \infty)$. Also, for the sake of simplicity, let the material coordinates x_i of the particle in the reference configuration be rectangular cartesian. Then the strain–displacement relations are

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) \tag{3.1}$$

where the comma denotes partial differentiation with respect to x_i .

The local forms of the laws of balance of linear momentum and angular momentum are

$$[\sigma^{ij}(\delta_{mj} + u_{m,j})]_{,i} + \rho F_m = \rho \ddot{u}_m \tag{3.2a}$$

$$\sigma^{ij} = \sigma^{ji} \tag{3.2b}$$

where the superposed dot indicates partial differentiation with respect to time and $\rho(\mathbf{x})$ is the mass density of the solid in the reference configuration. Here σ^{ij} are the contravariant components of the second Piola–Kirchhoff stress tensor measuring stress per unit area of the “undeformed” body, i.e. per unit area of the material surface in the reference configuration; and δ_{ij} denotes the Kronecker delta.

Since in this development we consider a class of materials whose response is not influenced by temperature and for which stress $\boldsymbol{\sigma}$ at time t is determined by the strain history $\boldsymbol{\gamma}^t(s)$, the stress is assumed to be given by the constitutive equation of the form

$$\sigma^{ij} = \mathcal{F}^{ij}[\boldsymbol{\gamma}^t(s)] \tag{3.3}$$

where $\boldsymbol{\gamma}^t(s) = \boldsymbol{\gamma}(t-s)$ is the strain $\boldsymbol{\gamma}$ at time $t-s$, $0 \leq s < \infty$ and \mathcal{F}^{ij} is a tensor-valued functional mapping $\boldsymbol{\gamma}^t(s)$ into $\boldsymbol{\sigma}(t)$. In the case of simple materials [10, 11], it is customary to assume the existence of a free energy density φ which is given by a constitutive equation

$$\varphi = \Phi[\boldsymbol{\gamma}^t_r; \boldsymbol{\gamma}(t)]. \tag{3.4}$$

Here we have decomposed the total strain history $\boldsymbol{\gamma}^t(s) = \boldsymbol{\gamma}(t-s)$, $s \in [0, \infty)$ into the “past history” $\boldsymbol{\gamma}^t_r(s) = \boldsymbol{\gamma}^t(s)$, $s \in (0, \infty)$ and the current strain, $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}^t(0)$. Then if $\Phi_{s=0}^\infty[\boldsymbol{\gamma}^t_r(s); \boldsymbol{\gamma}(t)]$ possesses sufficient smoothness properties,

$$\sigma^{ij} = \mathcal{F}^{ij}[\boldsymbol{\gamma}^t_r; \boldsymbol{\gamma}] = \rho \partial_{\boldsymbol{\gamma}} \Phi[\boldsymbol{\gamma}^t_r; \boldsymbol{\gamma}]. \tag{3.5}$$

Likewise, the internal dissipation is given by

$$\hat{\sigma} = \mathcal{D}[\boldsymbol{\gamma}^t_r; \boldsymbol{\gamma}] = \delta_{\boldsymbol{\gamma}} \Phi[\boldsymbol{\gamma}^t_r; \boldsymbol{\gamma} \dot{\boldsymbol{\gamma}}^t_r] \tag{3.6}$$

wherein $\delta_{\boldsymbol{\gamma}} \Phi_{s=0}^\infty[; \cdot]$ denotes the Frechet differential which is linear in $\dot{\boldsymbol{\gamma}}^t_r$, and $\partial_{\boldsymbol{\gamma}}$ is the ordinary partial differentiation of Φ with respect to $\boldsymbol{\gamma}$.

To complete the formulation of the field equations, the initial and boundary conditions must be added to (3.1)–(3.6). Let \mathcal{B}_u and \mathcal{B}_σ denote disjoint sets whose union is \mathcal{B} and let

the outward unit normal vector to \mathcal{B} be \mathbf{n} . Here \mathcal{B}_u is taken as the portion of the boundary where displacements are prescribed, whereas \mathcal{B}_s is the portion over which tractions are prescribed. Therefore, the displacement boundary conditions are

$$u_i = \hat{u}_i \quad \text{on } \mathcal{B}_u \times (-\infty, \infty) \tag{3.7a}$$

and the traction boundary conditions

$$T^i = n_j \sigma^{jm} (\delta_{mi} + u_{m,i}) = \hat{T}^i \quad \text{on } \mathcal{B}_s \times (-\infty, \infty). \tag{3.7b}$$

In (3.7), \hat{u}_i and \hat{T}^i are the prescribed functions of \mathbf{x} and t . Mixed conditions for displacements and tractions could also be included.

The associated initial conditions are

$$u_i(\mathbf{x}, 0) = d_i(\mathbf{x}) \tag{3.8a}$$

$$\dot{u}_i(\mathbf{x}, 0) = v_i(\mathbf{x}) \tag{3.8b}$$

where $\mathbf{d}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ are, respectively, the prescribed initial displacements and initial velocities.

In summary, the behavior of a nonlinearly viscoelastic body is described by conditions (3.7) and (3.8) and the system of equations:

$$g^*[\sigma^{ij}(\delta_{mj} + u_{m,j})]_{,i} + \rho f_m - \rho u_m = 0 \tag{3.9a}$$

$$\gamma_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) = 0 \tag{3.9b}$$

$$\sigma^{ij} - \int_{s=0}^{\infty} \mathcal{F}^{ij}[\gamma_r^t; \gamma] = 0 \tag{3.9c}$$

$$\hat{\sigma} - \int_{s=0}^{\infty} \mathcal{D}[\gamma_r^t; \gamma] = 0 \tag{3.9d}$$

wherein $g(t) = t$ and

$$f_m(\mathbf{x}, t) = [g^*F_m](\mathbf{x}, t) + \rho(\mathbf{x})[tv_m(\mathbf{x}) + d_m(\mathbf{x})]. \tag{3.10}$$

Here the functions \mathbf{f} are determined completely by the knowledge of the body forces and the initial history.

The symbol $*$ in (3.9a) denotes the convolution operator; i.e. if u and v are the scalar-valued functions defined on $\bar{V} \times (-\infty, \infty)$, then following Gurtin [1, 2], the convolution $u*v$ is the function on $\bar{V} \times [0, \infty)$ defined by

$$u*v = \int_0^t [u(\mathbf{x}, t-\tau)v(\mathbf{x}, \tau)] d\tau \tag{3.11}$$

$(\mathbf{x}, t) \in \bar{V} \times [0, \infty)$ and in (3.9a) convolution operation has been performed by taking the Laplace transform, re-arranging the terms and taking the inverse transform. Then the bilinear mapping of (2.6) can be defined by

$$\langle u, v \rangle = \int_V [u*v] dV \tag{3.12}$$

which satisfies the condition that if $\langle u, v \rangle = 0$, either $u = 0$ or $v = 0$ for $t \geq 0$. An alternative to (3.11) is to use Stieltjes integral

$$u^*dv = \int_{-\infty}^t u(\mathbf{x}, t - \tau) dv(\mathbf{x}, \tau) \tag{3.13}$$

provided this integral is meaningful; and $(\mathbf{x}, t) \in \bar{V} \times (-\infty, \infty)$. Then the bilinear map is

$$\langle u, dv \rangle = \int_V [u^*dv] dV. \tag{3.14}$$

4. VARIATIONAL PRINCIPLES FOR NONLINEAR VISCOELASTICITY

To construct a variational principle associated with the nonlinear theory of viscoelasticity described by (3.7)–(3.9), we follow a procedure adopted by Oden [9, 12] and rewrite the field equations (3.9) in the form

$$\hat{\mathcal{P}}(\Lambda) = \mathcal{P}(\Lambda) - \Gamma = \mathbf{0} \tag{4.1a}$$

or equivalently

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} & \mathcal{P}_{13} & \mathcal{P}_{14} \\ \mathcal{P}_{21} & \mathcal{P}_{22} & \mathcal{P}_{23} & \mathcal{P}_{24} \\ \mathcal{P}_{31} & \mathcal{P}_{32} & \mathcal{P}_{33} & \mathcal{P}_{34} \\ \mathcal{P}_{41} & \mathcal{P}_{42} & \mathcal{P}_{43} & \mathcal{P}_{44} \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{bmatrix} - \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \end{bmatrix} = \mathbf{0} \tag{4.1b}$$

wherein

$$\begin{aligned} \mathcal{P}_{11} &= \rho \delta_{im}; \mathcal{P}_{13} = -g^* \left[\frac{1}{2} \left(\delta_{mj} + \frac{\partial u_m}{\partial x_j} \right) \left(\delta_{in} \frac{\partial}{\partial x_n} + \delta_{jk} \frac{\partial}{\partial x_k} \right) + \frac{\partial^2 u_m}{\partial x_i \partial x_j} \right] \\ \mathcal{P}_{23} = \mathcal{P}_{32} = \mathcal{P}_{44} &= -g^*; \mathcal{P}_{31} = g^* \left[\frac{1}{2} \left(\delta_{ik} \frac{\partial}{\partial x_l} + \delta_{il} \frac{\partial}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \frac{\partial}{\partial x_l} \right) \right] \\ \mathcal{P}_{12} = \mathcal{P}_{14} = \mathcal{P}_{21} = \mathcal{P}_{22} = \mathcal{P}_{24} = \mathcal{P}_{33} = \mathcal{P}_{34} = \mathcal{P}_{41} = \mathcal{P}_{42} = \mathcal{P}_{43} &= 0 \end{aligned} \tag{4.2a}$$

and

$$\Lambda = \{u_i, \gamma_{kl}, \sigma^{ij}, \hat{\sigma}\}^T \tag{4.2b}$$

$$\Gamma = \{\rho f_m, -g^* \mathcal{F}^{ij} [;], 0, -g^* \mathcal{D} [;]\}^T. \tag{4.2c}$$

The operator $\mathcal{P}(\Lambda)$ is obviously the nonlinear operator consistent with (3.9).

In the application that follows, for the sake of conciseness, we shall assume homogeneous boundary conditions. Non-homogeneous boundary conditions can easily be introduced by additional terms in the respective functionals. Then making use of the

definition (3.12) for the bilinear mapping $\langle u, v \rangle$, it follows that in this case,

$$\begin{aligned} \langle \hat{\mathcal{P}}(s\Lambda), \Lambda \rangle &= \langle \mathcal{P}(s\Lambda) - \Gamma, \Lambda \rangle \\ &= \int_V \{ su_m^* \rho u_m - u_m^* g^* [s\sigma^{ij}(\delta_{mj} + su_{m,j})]_{,i} - u_m^* \rho f_m \\ &\quad - s\gamma_{ij}^* g^* \sigma^{ij} + \gamma_{ij}^* g^* \mathcal{F}_{s=0}^{ij} [\gamma_r^t; \gamma] - s\sigma^{ij} g^* \gamma_{ij} \\ &\quad + \sigma^{ij} g^* \frac{1}{2} (su_{i,j} + su_{j,i} + s^2 u_{m,i} u_{m,j}) - s\hat{\sigma}^* g^* \hat{\sigma} \\ &\quad + \hat{\sigma}^* g^* \mathcal{D}_{s=0} [\gamma_r^t; \gamma] \} ds dV. \end{aligned} \tag{4.3}$$

Thus, introducing (4.3) into (2.8), integrating by parts and making use of homogeneous boundary conditions and the commutativity and associativity of the convolution operator $[u^*v = v^*u, u^*(v^*w) = (u^*v)^*w]$, we obtain the functional

$$\begin{aligned} K(\Lambda) &= \frac{1}{2} \int_V [\rho u_m^* u_m - 2g^* u_m^* \sigma_{,i}^{im} - 2g^* \gamma_{im}^* \sigma^{im} + 2g^* \gamma_{im}^* \mathcal{F}_{s=0}^{im} [\gamma_r^t; \gamma] \\ &\quad - g^* \hat{\sigma}^* \hat{\sigma} + 2g^* \hat{\sigma}^* \mathcal{D}_{s=0} [\gamma_r^t; \gamma] - g^* u_m^* (\sigma^{ij} u_{m,j})_{,i} - 2\rho u_m^* f_m] dV. \end{aligned} \tag{4.4}$$

Note that in arriving (4.4) from (2.8), we have set $K_0 = 0$ and $\Lambda_0 = \mathbf{0}$.

Hence, we have derived a variational principle associated with the nonlinear operator equations (3.9); that is, $K(\Lambda)$ of (4.4) assumes a stationary value when Λ satisfies (3.9). We can set forth this result in the following theorem.

Theorem 2

Let $u_i(\mathbf{x}, t)$, $\gamma_{ij}(\mathbf{x}, t)$, $\sigma^{ij}(\mathbf{x}, t)$ and $\hat{\sigma}(\mathbf{x}, t)$, all defined for $(\mathbf{x}, t) \in \bar{V} \times (-\infty, \infty)$, satisfy the field equations of nonlinear viscoelasticity given by (3.9) with homogeneous boundary conditions. Then the functional $K(\Lambda)$ of (4.4) assumes a stationary value at $\Lambda = \{\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\sigma}, \hat{\sigma}\}$.

Proof. Let $\bar{\Lambda} = \{\bar{\mathbf{u}}, \bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\sigma}}, \bar{\hat{\sigma}}\}$ denote an arbitrary element in the domain of the operator \mathcal{P} . Then by (2.4) or (2.5), we have

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [K(\Lambda + \alpha \bar{\Lambda}) - K(\Lambda)] &= \frac{\partial}{\partial \alpha} K(\Lambda + \alpha \bar{\Lambda})|_{\alpha=0} \\ &= \int_V \left\{ (\rho u_m - g^* [\sigma^{ij}(\delta_{mj} + u_{m,j})]_{,i} - \rho f_m)^* \bar{u}_m \right. \\ &\quad + g^* (\frac{1}{2} [u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}] - \gamma_{ij})^* \bar{\sigma}^{ij} \\ &\quad \left. + g^* \left(\mathcal{F}_{s=0}^{ij} [\gamma_r^t; \gamma] - \sigma^{ij} \right)^* \gamma_{ij} + g^* \left(\mathcal{D}_{s=0} [\gamma_r^t; \gamma] - \hat{\sigma} \right)^* \bar{\hat{\sigma}} \right\} dV \\ &= \langle \hat{\mathcal{P}}(\Lambda), \bar{\Lambda} \rangle. \end{aligned}$$

That is, $\hat{\mathcal{P}}(\Lambda) = \text{grad } K(\Lambda)$. This proves that the functional $K(\Lambda)$ of (4.4) assumes a stationary value when Λ satisfies (3.9) and that (3.9) are, indeed, the Euler's equations of (4.4).

In the interest of brevity, we have considered homogeneous boundary conditions in arriving at (4.4). However, inclusion of non-homogeneous boundary conditions (3.7) into the basic functional (2.8) will result in simply additional terms appearing in (4.4), i.e. then we must add to the functional K of (4.4) the following functional :

$$K_1 = \frac{1}{2} \int_{\mathcal{B}_s} \{u_m^* g^* [n_1 \sigma^{ij} (2\delta_{mj} + u_{m,j}) - 2\hat{T}^m]\} d\mathcal{B} + \int_{\mathcal{B}_u} \{\hat{u}_m^* g^* T^m\} d\mathcal{B}. \tag{4.5}$$

We observe that the conditions of prescribed tractions on the portion \mathcal{B}_s and of the prescribed displacement on the portion \mathcal{B}_u of the boundary of the body, are determined by the application of (2.5) to (4.5); i.e. by varying u_i and $u_{i,j}$ on \mathcal{B}_s and T^m on \mathcal{B}_u , then we find that the boundary conditions (3.7) as well as (4.1a) are satisfied at critical points of $K + K_1$.

5. ALTERNATE PRINCIPLES

In this section we cite a number of alternate principles and demonstrate that, under appropriate additional assumptions, the existing variational principles for problems in linear viscoelasticity [2, 3, 5, 6] can be derived from the general principles developed in this paper.

Firstly, we delete the internal dissipation equation (3.9d) from the list of field equations and, using (4.4), obtain the new functional

$$K(\mathbf{S}) = \frac{1}{2} \int_V \left\{ \rho u_m^* u_m - 2g^* u_m^* \sigma_{,i}^{im} - 2g^* \gamma_{im}^* \sigma^{im} + 2g^* \gamma_{im}^* \overset{\infty}{\mathcal{F}}^{im} [\gamma_r^t; \gamma] - g^* u_m^* (\sigma^{ij} u_{m,j})_{,i} - 2\rho u_m^* f_m \right\} dV. \tag{5.1}$$

If $\mathbf{S} = \{\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\sigma}\}$ is the solution to the problem (3.9a)–(3.9c), then $K(\mathbf{S})$ in (5.1) assumes a stationary value at \mathbf{S} , i.e. by setting $\delta K(\mathbf{S}) = \mathbf{0}$, we obtain (3.9a)–(3.9c) as the Euler’s equations of (5.1). Integrating (5.1) by parts and making use of the homogeneous boundary conditions, the functional $K(\mathbf{S})$ of (5.1) can be written in an alternate form

$$K(\mathbf{S}) = \frac{1}{2} \int_V \left\{ \rho u_i^* u_i + 2g^* \gamma_{ij}^* \overset{\infty}{\mathcal{F}}^{ij} [\gamma_r^t; \gamma] + g^* [u_{i,j} + u_{j,i} + u_{m,i} u_{m,j} - 2\gamma_{ij}]^* \sigma^{ij} - 2\rho u_i^* f_i \right\} dV. \tag{5.2}$$

Now consider the case when \mathbf{u} and $\boldsymbol{\gamma}$ of $\mathbf{S} = \{\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\sigma}\}$ satisfy the strain–displacement relation (3.9b). Then $K(\mathbf{S})$ given by (5.2) reduces to $J(\mathbf{S})$, where

$$J(\mathbf{S}) = \frac{1}{2} \int_V \left\{ \rho u_i^* u_i + 2g^* \gamma_{ij}^* \overset{\infty}{\mathcal{F}}^{ij} [\gamma_r^t; \gamma] - 2\rho u_i^* f_i \right\} dV. \tag{5.3}$$

For the special case in which $\overset{\infty}{\mathcal{F}}^{ij} [;]$ is linear in the strain histories, (5.3) is identical with (4.6) of [3] for homogeneous boundary conditions.

For the case of quasi-static motion, (5.3) reduces to

$$\bar{J}(\mathbf{S}) = \int_V [\gamma_{ij} \sigma^{ij} - \rho u_i F_i] dV. \tag{5.4}$$

Then (5.4) together with (3.1), (3.11), (2.4), and the divergence theorem implies that

$$\delta_{\mathbf{u}}\bar{J}(\mathbf{S}) = - \int_V \{ [\sigma^{ij}(\delta_{mj} + u_{m,j})]_{,i} + \rho F_m \} \delta u_m \, dV. \tag{5.5}$$

If $\delta_{\mathbf{u}}\bar{J}(\mathbf{S})$ vanishes, the equilibrium equations (3.2a) are satisfied. Hence (5.4) describes a form the principle of stationary potential energy that appears in [1] for homogeneous boundary conditions.

It is also interesting to cast certain of the functionals in terms of the free energy functional Φ of (3.4). If we consider the case of quasi-static motion and define the functional

$$H(\mathbf{u}) = \int_V \left\{ \int_{s=0}^{\infty} \Phi [\gamma'_r(\mathbf{u}); \gamma(\mathbf{u})] - \hat{\sigma} - \rho F_m u_m \right\} \, dV \tag{5.6}$$

then

$$\begin{aligned} \delta_{\mathbf{u}}H(\mathbf{u}) = \int_V \left\{ \partial_{\gamma} \int_{s=0}^{\infty} \Phi [\gamma'_r(\mathbf{u}); \gamma(\mathbf{u})] (\delta_{mj} + u_{m,j}) \delta u_{m,i} + \delta_{\gamma} \int_{s=0}^{\infty} \left[\gamma'_r(\mathbf{u}); \gamma(\mathbf{u}) \right] \frac{\partial \gamma}{\partial u_i} \dot{u}_i \right. \\ \left. - \delta_{\mathbf{u}}\hat{\sigma} - \rho F_m \delta u_m \right\} \, dV. \end{aligned} \tag{5.7}$$

Since

$$\delta_{\mathbf{u}}\hat{\sigma} = \delta_{\gamma} \int_{s=0}^{\infty} \left[\gamma'_r(\mathbf{u}); \gamma(\mathbf{u}) \right] \frac{\partial \gamma}{\partial u_i} \dot{u}_i \tag{5.8}$$

we obtain, with the aid of (3.6), the divergence theorem, and the homogeneous boundary conditions,

$$\delta_{\mathbf{u}}H(\mathbf{u}) = - \int_V \{ [\sigma^{ij}(\delta_{mj} + u_{m,j})]_{,i} + \rho F_m \} \delta u_m \, dV. \tag{5.9}$$

This obviously vanishes for arbitrary displacements provided the equilibrium equations are satisfied. Thus the functional $H(\mathbf{u})$ is again a generalization of the principle of stationary potential energy. Indeed, if all histories are the rest histories and $\hat{\sigma} = 0$, Φ can be associated with the strain energy and (5.6) becomes precisely the potential energy of an elastic body.

We now consider the quasi-static motion of linear viscoelastic solid. The internal dissipation $\hat{\sigma}$ in (3.9d) is neglected as a second-order quantity and the linearized equations (3.9a)–(3.9c) are then written in the form

$$\begin{aligned} \sigma^ij + \rho f_i &= 0 \\ \sigma^ij - E^{ijmn} \mathbf{d}\gamma_{mn} &= 0 \\ \gamma_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) &= 0. \end{aligned} \tag{5.10}$$

We follow the procedure adopted in Section 4, and set

$$\tilde{\mathbf{A}} = \begin{Bmatrix} \mathbf{d}u_i \\ \mathbf{d}\gamma_{ij} \\ \sigma^{ij} \end{Bmatrix}; \quad \tilde{\mathbf{\Gamma}} = \begin{Bmatrix} -\rho f_i \\ 0 \\ 0 \end{Bmatrix}. \tag{5.11}$$

In this case, we use the definition (3.13) for $\langle u, v \rangle$ and obtain

$$\bar{K}(\bar{\mathbf{A}}) = \frac{1}{2} \int_V \{d\gamma_{ij} * E^{ijmn} * d\gamma_{mn} - 2 du_i * \sigma_j^{ij} - 2 d\gamma_{ij} * \sigma^{ij} - 2 du_i * \rho f_i\} dV. \quad (5.12)$$

Then $\bar{K}(\bar{\mathbf{A}})$ of (5.12) is the functional governing the variational statement of (5.10) and is identical with (49) of [5] and (3.1) of [2].

Various other alternate principles can also be derived from the general principles developed herein.

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Абстракт—Приводится общий вариационный принцип динамического поведения нелинейных вязкоэластичных твердых веществ. Также приводятся несколько альтернативных принципов.